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2001 J. Phys. A: Math. Gen. 34 9815

(http://iopscience.iop.org/0305-4470/34/46/305)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 9815–9825 PII: S0305-4470(01)24646-0

Determination of diffusion kernel on fractals

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Received 9 May 2001
Published 9 November 2001
Online at stacks.iop.org/JPhysA/34/9815

Abstract
When the diffusion set is a fractal, whether the generating mappings of the diffusion set are linear or nonlinear, the concrete expressions for the diffusion kernel are obtained. Moreover, the fractional diffusion exponent γ of the diffusion kernel must satisfy $0 < \gamma < 1$. In addition, the inverse problem of the above case is also discussed.

PACS numbers: 05.45.Df, 05.40.Fb, 05.60.-k

1. Introduction

In recent years the phenomenon of unusual transport properties on fractal structures has attracted more and more attention [1–7]. Much work has focused on understanding diffusion processes on such spatially correlated media. Regarding diffusion, the probability density $P(r,t)$, which is the probability of finding a random walker at time t at a distance r from its starting point at $t = 0$, plays a central role. Diffusion on fractals exhibits many anomalous features due to the geometrical constraints imposed by the complex structure on the diffusion process. More important analytical calculations (supported by extensive exact enumeration results) show that on a large class of fractal structures the probability $P(r,t)$ has asymptotically a non-Gaussian shape of the form ([7] and references therein)

$$P(r,t) \sim t^{-d_f/d_w} \exp[-\mathrm{const}(r/R)^u] \tag{1.1}$$

when $r/R \gg 1$ and $t \to \infty$, where $u = \frac{d_w}{d_w - 1}$, d_w is anomalous diffusion exponent of diffusion set, and d_f is the fractal dimension. Since $1 < u < 2$, the shape described (1.1) is called a stretched Gaussian [7] or [8]. How to give a diffusion equation which correctly describes the above non-Gaussian behaviour of $P(r,t)$ is still an open problem [7]. In [7, 8], a new type of fractional diffusion equation whose asymptotic solution coincides with the result (1.1) is proposed and a more general conservation equation containing an explicit reference of the diffusion process at previous times is introduced. Therefore, an integral relation of the form

$$j(r,t) = \int_0^t \mathrm{i}(r,\tau)\,\mathrm{d}\tau = r^{d_f-1}\int_0^t K(t,\tau)P(r,\tau)\,\mathrm{d}\tau = r^{d_f-1}\int_0^t K(t-\tau)P(r,\tau)\,\mathrm{d}\tau \quad (1.2)$$

and the constitutive local equation on fractals

$$\mathrm{i}(r,t) = -Br^{d_f-1}r^{-\theta'}\left(\frac{\partial P(r,t)}{\partial r} + \frac{\mathcal{K}}{r}P(r,t)\right) \quad (1.3)$$

are proposed in [7,8], where $\mathrm{i}(r,t)$ is the radial probability current, $K(t,\tau) = K(t-\tau)$ is the diffusion kernel, $B > 0$ is the diffusion coefficient, $\theta' > 0$, and $\mathcal{K}$ remain to be determined. Moreover, under the assumption that on fractals the diffusion kernel should behave as

$$K(t-\tau) = (t-\tau)^{-\gamma} \qquad 0 < \gamma < 1 \quad (1.4)$$

and the normalization condition $\int_0^\infty r^{d_f-1}P(r,t)\,\mathrm{d}t = 1$ the asymptotic solution of the fractional diffusion equation induced by (1.2) and (1.3) is obtained in [7] or [8].

But they did not show what relation between the exponents γ in (1.4) and fractals is and how γ is determined. Recently, in a previous paper [9], for some fractals we gave the expressions of the diffusion exponent γ and the diffusion kernel $K(t)$. However, determination of the constant in the expression of $K(t)$ is still an open problem.

In this paper we will give the expression of the unknown constant in the expression of $K(t)$ and the exact approximation of diffusion kernels on fractals. In addition, we also discuss the inverse problem of the above case.

2. Determination of diffusion kernel

For any given $T \in (0,\infty)$ and $k \in \{2,3,\ldots\}$, let $E_0 = [0,T]$ and let $\{\varphi_j(x)\}_{j=1}^k$ be the contractive transformations on E_0, i.e.

$$|\varphi_j(x) - \varphi_j(y)| \leqslant c_j|x-y| \qquad \forall x,y \in E_0 \qquad 0 < c_j < 1. \quad (2.1)$$

Suppose $\varphi_j(E_0) \cap \varphi_i(E_0) = \emptyset$, $i \neq j$. Then the non-empty compact set

$$E_T = \bigcap_{n=1}^{\infty} E(n) \quad (2.2)$$

is called a Cantor type set [9], where

$$E(n) = \bigcup_{j_1,\ldots,j_n} E_{j_1\cdots j_n} \qquad E_{j_1\cdots j_n} = \varphi_{j_1}\circ\cdots\circ\varphi_{j_n}(E_0).$$

For any given probability vector $\boldsymbol{p} = (p_1, p_2, \ldots, p_k)$ (i.e. $p_i > 0$, $\sum_{i=1}^k p_i = 1$), let the measure μ_n on $E(n)$ be a probability measure defined by $\mathrm{d}\mu_n(\tau) = m_n(\tau)\,\mathrm{d}\tau$:

$$m_n(\tau) = \sum_{j_1,\ldots,j_n\in I} p_{j_1}\cdots p_{j_n}\frac{\chi_{E_{j_1\cdots j_n}}(\tau)}{|E_{j_1\cdots j_n}|} \qquad I = \{1,\ldots,k\}$$

where

$$\chi_{E_{j_1\cdots j_n}}(\tau) = \begin{cases} 1 & \tau \in E_{j_1\cdots j_n} \\ 0 & \text{otherwise.} \end{cases}$$

Then, as shown in [10, 11], μ_n weakly converges to a unique probability measure μ as $n \to +\infty$ such that $\int_{E_T}\mathrm{d}\mu(\tau) = 1$, $\mathrm{supp}(\mu) = E_T$, and

$$\mu(\cdot) = \sum_{j=1}^k p_j\mu\circ\varphi_j^{-1}(\cdot). \quad (2.3)$$

Let E_T be a diffusion set. The measure μ is called the *diffusion measure*. Thus we have the conservation equation

$$\int_0^t \mathrm{i}(r,\tau)\,\mathrm{d}\tau = r^{d_f-1}\int_0^t P(r,\tau)\,\mathrm{d}\mu(t-\tau) \tag{2.4}$$

where d_f denote the fractal dimension of E_T. Let

$$M(p) = \int_0^{+\infty} \mathrm{e}^{-p\tau}\,\mathrm{d}\mu(\tau) \tag{2.5}$$

be the Laplace transform of the measure $\mu(\tau)$. Since μ is supported on $E_T \subset E_0 = [0, T]$, $\mu \circ \varphi_j^{-1}(\cdot)$ is supported on $\varphi_j(E_T) \subset E_j$. Acting the Laplace transformation on both sides of (2.3) we have

$$M(p) = \sum_{j=1}^{k} p_j \int_0^T \mathrm{e}^{-p\varphi_j(\tau)}\,\mathrm{d}\mu(\tau). \tag{2.6}$$

Assume that, for each $j \in I$, $\varphi_j(x)$ is a similarity or $\varphi_j(x)$ is a $C^{1+\alpha}$-differentiable mapping ($\varphi_j(x)$ is differentiable with a Hölder continuous derivative φ_j' satisfying $|\varphi_j'(x) - \varphi_j'(y)| < \beta_j|x-y|^\alpha$, where $\alpha > 0$ and $\beta_j > 0$ are constants).

Let

$$b_j = \varphi_j(0) \qquad \xi_j = \varphi_j'(0) \quad 0 < \xi_j < 1 \quad j = 1, \ldots, k \tag{2.7}$$

and $0 = b_1 < b_2 < \cdots < b_k < \varphi_k(T) = T$. Let $\tilde{\varphi}_j(x) = \varphi_j(x) - b_j$, then

$$\tilde{\varphi}_j(x) = \xi_j x + \mathrm{O}(x^{1+\alpha}) \qquad \text{as} \quad x \to 0. \tag{2.8}$$

Thus

$$\int_0^T \mathrm{e}^{-p\varphi_j(\tau)}\,\mathrm{d}\mu(\tau) = \mathrm{e}^{-b_j p}\int_0^T \exp(-p\tilde{\varphi}_j(\tau))\,\mathrm{d}\mu(\tau) = \mathrm{e}^{-b_j p} M(\xi_j p) + \mathrm{o}(1) \tag{2.9}$$

as $\mathrm{Re}\, p \to +\infty$ (cf [12]). It follows from (2.9) and (2.6) that

$$M(p) = \sum_{j=1}^{k} p_j \mathrm{e}^{-b_j p} M(\xi_j p) + \mathrm{o}(1) \tag{2.10}$$

as $\mathrm{Re}\, p \to +\infty$.

If $\varphi_1(0) = 0$, from (2.10) we have

$$M(p) = p_1 M(\xi_1 p) + \mathrm{o}(1) \qquad (\text{as Re } p \to +\infty) \tag{2.11}$$

(cf [11, 13] or [12, 13]). Solving the function equation

$$\overline{M}(p) = p_1 \overline{M}(\xi_1 p)$$

we have

$$\overline{M}(p) = A p^{-\nu} \tag{2.12}$$

where A is a constant and $\nu = \ln p_1 / \ln \xi_1$.

Because $M(0) = \int_0^{+\infty} \mathrm{d}\mu(\tau) = \int_0^T \mathrm{d}\mu(\tau) = 1$, but $\overline{M}(0) \neq 0$, so we choose an approximation $\tilde{M}(p)$ of $M(p)$:

$$M(p) \simeq \tilde{M}(p) = A' p^{-\nu}\big[1 - \exp(-p^\nu/A')\big]. \tag{2.13}$$

It is easy to see that $\tilde{M}(0) = \lim_{p\to 0} \tilde{M}(p) = 1$.

Applying the theory of complex analysis we can obtain the inverse Laplace transform $\mathcal{L}^{-1}[\tilde{M}(p)]$ of $\tilde{M}(p)$:

$$\begin{aligned}\mathcal{L}^{-1}[\tilde{M}(p)] &= \frac{1}{2\pi\mathrm{i}}\int_{a-i\infty}^{a+i\infty} \mathrm{e}^{pt}\tilde{M}(p)\,\mathrm{d}p \quad (a>0)\\ &= \frac{1}{2\pi\mathrm{i}}\int_{a-i\infty}^{a+i\infty} \mathrm{e}^{pt}A'p^{-\nu}\left(1-\exp(-p^{\nu}/A')\right)\,\mathrm{d}p\\ &= \frac{A'\Gamma(\gamma)\sin\nu\pi}{\pi}t^{-\gamma}\end{aligned} \tag{2.14}$$

if and only if $0<\gamma<1$, where $\gamma = 1-\nu = 1-\ln p_1/\ln\xi_1$, see the appendix.

It follows from (2.5), (2.13) and (2.14) that

$$\mathrm{d}\mu(t) \simeq K(t)\,\mathrm{d}t \tag{2.15}$$

$$K(t) = \frac{A'\Gamma(\gamma)\sin\nu\pi}{\pi}t^{-\gamma}. \tag{2.16}$$

where $K(t)$ is the diffusion kernel function. From $\int_0^T \mathrm{d}\mu(t) = 1$ and (2.16) we obtain that

$$\frac{A'\Gamma(\gamma)\sin\nu\pi}{\pi}\int_0^T t^{-\gamma}\,\mathrm{d}t = \frac{A'\Gamma(\gamma)\sin\nu\pi}{(1-\gamma)\pi}T^{1-\gamma} \simeq 1.$$

Therefore, we obtain the approximations of the constant A' in (2.16) and the diffusion kernel $K(t)$:

$$A' \simeq \frac{\nu\pi}{T^{\nu}\Gamma(1-\nu)\sin\nu\pi}. \tag{2.17}$$

$$K(t) \simeq \frac{\nu}{T^{\nu}}t^{\nu-1} = \frac{\nu}{T^{\nu}}t^{-\gamma} \tag{2.18}$$

where

$$\gamma = 1-\nu = 1-\ln p_1/\ln\xi_1 \tag{2.19}$$

and $0<\gamma<1$.

If we use $\tilde{M}(\xi_j p)$ as an approximation of $M(\xi_j p)$, i.e.

$$M(\xi_j p) \simeq \tilde{M}(\xi_j p) = A'(\xi_j p)^{-\nu}\left[1-\exp(-(\xi_j p)^{\nu}/A')\right] \tag{2.20}$$

as $\mathrm{Re}\ p\to+\infty$. Thus we have

$$M(p) \simeq \sum_{j=1}^{k} p_j\mathrm{e}^{-b_j p}A'(\xi_j p)^{-\nu}\left[1-\exp\left(-(\xi_j p)^{\nu}/A'\right)\right] \tag{2.21}$$

as $\mathrm{Re}\ p\to+\infty$.

Write $f(t) = \mathcal{L}^{-1}(M(p))$. Applying the properties of the Laplace transform we obtain

$$\mathcal{L}^{-1}(\mathrm{e}^{-b_j p}M(\xi_j p)) = \frac{1}{\xi_j}f(t-b_j/\xi_j)\eta(t-b_j) \tag{2.22}$$

where $\eta(x)=0$ for $x<0$ and 1 for $x>0$. It follows from (2.14) and (2.22) that

$$\begin{aligned}&\mathcal{L}^{-1}\left(\sum_{j=1}^{k} p_j\mathrm{e}^{-b_j p}A'(\xi_j p)^{-\nu}(1-\exp(-(\xi_j p)^{\nu}/A'))\right)\\ &\quad= \frac{A'\Gamma(1-\nu)\sin\nu\pi}{\pi}\left(p_1\xi_1^{-\nu}t^{-\gamma} + \sum_{j=2}^{k} p_j\xi_j^{-\nu}(t-b_j)^{-\gamma}\eta(t-b_j)\right)\end{aligned} \tag{2.23}$$

if and only if $0 < \gamma < 1$, where $\gamma = 1 - \nu$ and $\nu = \ln p_1 / \ln \xi_1$. That is,

$$d\mu(t) \simeq \frac{A'\Gamma(1-\nu)\sin\nu\pi}{\pi}\left(p_1\xi_1^{-\nu}t^{-\gamma} + \sum_{j=2}^{k} p_j\xi_j^{-\nu}(t-b_j)^{\gamma}\eta(t-b_j)\right)dt \tag{2.24}$$

if and only if $0 < \gamma < 1$. It follows from $\int_0^T d\mu(t) = 1$ that

$$A' \simeq \frac{\pi\nu}{\Gamma(1-\nu)[p_1\xi_1^{-\nu}T^{\nu} + \sum_{j=2}^{k} p_j\xi_j^{-\nu}(T-b_j)^{\nu}]\sin\nu\pi}. \tag{2.25}$$

Therefore, we obtain the further approximation of the diffusion kernel $K(t)$

$$K(t) \simeq \frac{\nu\big(p_1\xi_1^{-\nu}t^{-\gamma} + \sum_{j=2}^{k} p_j\xi_j^{-\nu}(t-b_j)^{-\gamma}\eta(t-b_j)\big)}{p_1\xi_1^{-\nu}T^{\nu} + \sum_{j=2}^{k} p_j\xi_j^{-\nu}(T-b_j)^{\nu}} \tag{2.26}$$

where

$$\gamma = 1 - \nu \qquad 0 < \nu = \ln p_1/\ln\xi_1 < 1. \tag{2.27}$$

3. Inverse problem

From section 2 we can see that if a diffusion fractal and a probability vector $\boldsymbol{p} = (p_1, \dots, p_k)$ (which determines the self-similar measure) are given, then the diffusion kernel $K(t)$ is approximately determined and so we can deduce the expression of corresponding probability current $j(r, t)$. However, in general, the probability vector $\boldsymbol{p} = (p_1, \dots, p_k)$ is not known. Therefore, we need to discuss an inverse problem. That is, if $\int_0^T t^n d\mu(t)$, $t \in \{0, 1, \dots, k\}$, may be measured, how is the corresponding diffusion kernel determined?

We can assume the diffusion fractal is a Cantor type set. If not, it can be approximately determined by the way given in [14]. That is, we can find a finite set of contractive affine transformations φ_j, $j = 1, \dots, k$, with respective contractivity factors $s_1, \dots, s_k$ such that the transformation $\varphi(B) = \bigcup_{j=1}^k \varphi_j(B)$ with contractivity factor $s = \max\{s_j : j = 1, \dots, k\}$ exists a unique fixed point E (compact set) satisfying $E = \varphi(E) = \bigcup_{j=1}^k \varphi_j(E)$ and $E = \lim_{n\to\infty} \varphi^n(B)$ for any compact set B.

Therefore, for any given $T \in (0, +\infty)$, assume the diffusion fractal E_T is generated by a set of the contraction transformations $\varphi_j(x)$, $j = 1, \dots, k$, where $x \in [0, T]$ and when $x \ll 1$ $\varphi_j(x) = \varphi_j(0) + \xi_j x + O(1)$. We know that for any given $\{\varphi_j\}_{j=1}^k$ and $\{p_j\}_{j=1}^k$ there exists the unique self-similar measure

$$\mu(\cdot) = \sum_{j=1}^{k} p_j \mu \circ \varphi_j^{-1}(\cdot).$$

Let

$$M(p) = \int_0^{+\infty} e^{-p\tau}\, d\mu(\tau)$$

be the Laplace transform of the measure $\mu(\tau)$. Then

$$M(p) = \sum_{j=1}^{k} p_j e^{-\varphi_j(0)p} M(\xi_j p) + o(1).$$

When Re $p \gg 1$,

$$\int_0^T e^{-pt}\, d\mu(t) = \sum_{j=1}^{k} p_j \int_0^T e^{-p(\varphi_j(0)+\xi_j t)}\, d\mu(t) + o(1)$$

i.e.

$$\int_0^T \sum_{n=0}^{\infty} \frac{1}{n!}(-pt)^n \, \mathrm{d}\mu(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{j=1}^{k} p_j \int_0^T [p(\varphi_j(0) + \xi_j t)]^n \, \mathrm{d}\mu(t) + \mathrm{o}(1).$$

Thus

$$\int_0^T t^n \, \mathrm{d}\mu(t) = \sum_{j=1}^{k} p_j \int_0^T (\xi_j t + b_j)^n \, \mathrm{d}\mu(t) + \mathrm{o}(1) \qquad b_j = \varphi_j(0) \quad n = 0, 1, \ldots, k.$$

That is, $\boldsymbol{p} = (p_1, \ldots, p_k)$ should satisfy the equation

$$\int_0^T t^n \, \mathrm{d}\mu(t) = \sum_{j=1}^{k} p_j \int_0^T (\xi_j t + b_j)^n \, \mathrm{d}\mu(t) \qquad n = 0, 1, \ldots, k.$$

Let

$$j_n = \int_0^T t^n \, \mathrm{d}\mu(t) \qquad a_{nj} = \int_0^T (\xi_j t + b_j)^n \, \mathrm{d}\mu(t) \qquad j, n = 1, \ldots, k.$$

Then

$$\begin{aligned} &\sum_{j=1}^{k} p_j = 1 \\ &\sum_{j=1}^{k} a_{nj} p_j = j_n \qquad n = 1, \ldots, k \end{aligned} \tag{3.1}$$

where $a_{nj} = \sum_{i=0}^{n} C_i^n \xi_j^i b_j^{n-i} j_i$, $C_i^n = \frac{n(n-1)\cdots(n-i+1)}{i!}$.

If $j_n, n = 1, \ldots, k$, are known or can be measured, then the linear equations (3.1) are determined. Since the self-similar measure is unique, the corresponding probability vector $\boldsymbol{p} = (p_1, \ldots, p_k)$ is uniquely determined. Therefore, if the coefficient determinant

$$D_k = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ & \cdots & \cdots & \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{vmatrix} \neq 0$$

the unique solution of equations (3.1) is

$$p_j = \frac{\overline{D}_j}{D_k} \qquad j = 1, \ldots, k$$

where

$$\overline{D}_j = \begin{vmatrix} a_{11} & \cdots & a_{1(j-1)} & j_1 & a_{1(j+1)} & \cdots & a_{1k} \\ a_{21} & \cdots & a_{2(j-1)} & j_2 & a_{2(j+1)} & \cdots & a_{2k} \\ & \cdots & \cdots & & \cdots & & \\ a_{k1} & \cdots & a_{k(j-1)} & j_k & a_{k(j+1)} & \cdots & a_{kk} \end{vmatrix}.$$

Therefore, using formulae (2.27) and (2.26) in section 2 we can obtain the diffusion exponent γ and the kernel function $K(t)$.

4. Conclusions

(1) When the diffusion set is E_T, whether the generating mappings $\{\varphi_j(x)\}_{j=1}^{k}$ of the diffusion set are linear, nonlinear, increasing or decreasing and no matter which self-similar measure is taken, the diffusion exponent γ is always determined only by $1-\nu = 1 - \ln p_1 / \ln |\varphi_1'(0)|$,

and γ must satisfy $0 < \gamma < 1$, i.e. $p_1 > |\varphi_1'(0)|$, where p_1 is the first weight of self-similar measure defined on the diffusion set and $\varphi_1'(0)$ is the derivative at 0 of the first generating mapping of the diffusion set, but it does not depend on the other weights of self-similar measure and other mappings.

(2) The constant A' in the diffusion kernel expression $K(t)$ is approximately given by the formulae (2.17) or (2.25), it depends both on the generating mappings of the diffusion set and on the weights of the self-similar measure.

(3) From (2.26) the diffusion kernel $K(t)$ is determined not only by $\nu = \ln p_1 / \ln|\varphi_1'(0)|$, but also by all $\{p_j\}_{j=1}^k$, $\{\varphi_j(0)\}_{j=1}^k$, $\{\varphi_j'(0)\}_{j=1}^k$ and T. This means that the diffusion kernel $K(t)$ depends both on the generating mappings of the diffusion set and on the weights of the self-similar measure.

(4) $\gamma = 1 - d_f$ if and only if $p_1 = \xi_1^{d_f}$ and $\gamma = d_f$ if and only if $p_1 = \xi^{(1-d_f)}$.

(5) From the formula $\gamma = 1 - \ln p_1 / \ln \xi_1$, we can see that there is no direct relationship between the fractional diffusion exponent γ and the geometrical characteristics of the fractal structure of the diffusion set E_T considered. This is determined by the methodology.

(6) From section 3 if contractive transformations $\{\varphi_j(x)\}_{j=1}^k$ and $\int_0^T t^n \, \mathrm{d}\mu(t)$ are given, then the corresponding diffusion kernel and diffusion exponent can be uniquely determined.

Acknowledgments

Project supported by NSFC and Shanghai City Foundation of Selected Academic Research.

Appendix

The inverse Laplace transform of $\mathcal{L}^{-1}[\tilde{M}(p)]$ can be evaluated from the complex inversion formula

$$\mathcal{L}^{-1}[\tilde{M}(p)] = \frac{1}{2\pi\mathrm{i}} \int_{a-i\infty}^{a+i\infty} \mathrm{e}^{pt} \tilde{M}(p) \, \mathrm{d}p \tag{A.1}$$

following the integration path shown in figure A.1.

Applying the theorem of Cauchy (A.1) becomes

$$\begin{aligned}
\mathcal{L}^{-1}[\tilde{M}(p)] &= \mathcal{L}^{-1}\left[Ap^{-\nu}(1 - \exp(-p^\nu/A))\right] \\
&= \frac{1}{2\pi\mathrm{i}} \int_{a-i\infty}^{a+i\infty} A\mathrm{e}^{pt} p^{-\nu}(1 - \exp(-p^\nu/A)) \, \mathrm{d}p \\
&= -\lim_{\substack{R\to+\infty \\ \varepsilon\to+0}} \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_\varepsilon+\Gamma_1+\Gamma_2+C_1+C_2} A\mathrm{e}^{pt} p^{-\nu}(1 - \exp(-p^\nu/A)) \, \mathrm{d}p \\
&= -\lim_{\substack{R\to+\infty \\ \varepsilon\to+0}} \frac{1}{2\pi\mathrm{i}} (I_1 + I_2 + I_3 + I_4)
\end{aligned} \tag{A.2}$$

where

$$I_1 = \int_{\Gamma_\varepsilon} A\mathrm{e}^{pt} p^{-\nu}(1 - \exp(-p^\nu/A)) \, \mathrm{d}p \qquad I_2 = \int_{\Gamma_1} A\mathrm{e}^{pt} p^{-\nu}(1 - \exp(-p^\nu/A)) \, \mathrm{d}p,$$

$$I_3 = \int_{\Gamma_2} A\mathrm{e}^{pt} p^{-\nu}(1 - \exp(-p^\nu/A)) \, \mathrm{d}p \qquad I_4 = \int_{C_1+C_2} A\mathrm{e}^{pt} p^{-\nu}(1 - \exp(-p^\nu/A)) \, \mathrm{d}p.$$

We might as well assume $A > 0$. First, we prove that for $\nu > 0$

$$\lim_{\varepsilon\to+0} \left\{ I_1 = \int_{\Gamma_\varepsilon} A\mathrm{e}^{pt} p^{-\nu}(1 - \exp(-p^\nu/A)) \, \mathrm{d}p \right\} = 0 \tag{A.3}$$

where $\Gamma_\varepsilon : p = \varepsilon \mathrm{e}^{\mathrm{i}\theta}$, $-\pi < \theta < \pi$, $\mathrm{d}p = \mathrm{i}\varepsilon\mathrm{e}^{\mathrm{i}\theta} \, \mathrm{d}\theta$.

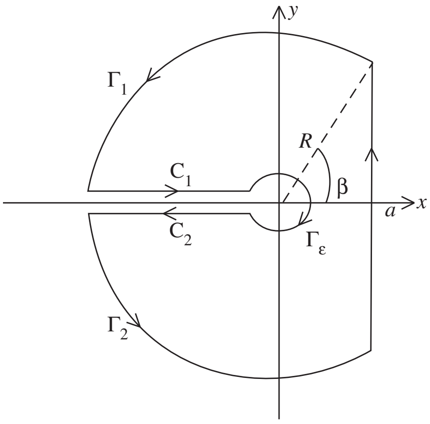

Figure A.1. Contour of integration in the complex plane for evaluating the inverse Laplace transform formula (A.1).

In fact,

$$
\begin{aligned}
|I_1| &\leqslant \int A\left|\mathrm{e}^{pt}p^{-\nu}\left(1-\exp(-p^\nu/A)\right)\right||\,\mathrm{d}p| \\
&\leqslant A\varepsilon^{1-\nu}\int_{-\pi}^{\pi}\mathrm{e}^{t\varepsilon\cos\theta}|z|\left[1+\frac{1}{2!}|z|+\cdots+\frac{1}{n!}|z|^{n-1}+\cdots\right]\mathrm{d}\theta \qquad (z=\varepsilon^\nu\mathrm{e}^{\mathrm{i}\theta\nu}/A) \\
&\leqslant 2\varepsilon^{1-\nu}\int_0^{\pi}\varepsilon^\nu(e-1)\exp(t\varepsilon\cos\theta)\,\mathrm{d}\theta < 2\pi(e-1)\varepsilon\mathrm{e}^{t\varepsilon}
\end{aligned}
$$

when $\varepsilon \ll 1$ and $\nu > 0$, because when $\varepsilon \ll 1$ and $\nu > 0$, $|z| = \varepsilon^\nu/A < 1$ and $1+\frac{1}{2!}|z|+\cdots+\frac{1}{n!}|z|^{n-1}+\cdots < e-1$. Therefore (A.3) holds.

Second, we prove that

$$\lim_{R\to+\infty}\left\{I_2=\int_{\Gamma_1}A\mathrm{e}^{pt}p^{-\nu}\left(1-\exp(-p^\nu/A)\right)\,\mathrm{d}p\right\}=0 \tag{A.4}$$

if and only if $0 < \nu < 1$. Here $\Gamma_1 : p = R\mathrm{e}^{\mathrm{i}\theta}$, $\beta < \theta < \pi$, $\mathrm{d}p = \mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta$.

Case I. $0 < \nu < 1$.

Note that

$$
\begin{aligned}
|I_2| &\leqslant A\int_{\Gamma_1}|p^{-\nu}|\left[|\exp(pt)|+|\exp(pt-p^\nu/A)|\right]|\,\mathrm{d}p| \\
&= AR^{1-\nu}\left[\int_\beta^\pi\exp(tR\cos\theta)\,\mathrm{d}\theta+\int_\beta^\pi\exp(tR\cos\theta-R^\nu\cos(\nu\theta)/A)\,\mathrm{d}\theta\right] \\
&= I_{21}+I_{22}
\end{aligned} \tag{A.5}
$$

where

$$
\begin{aligned}
I_{21} &= AR^{1-\nu}\int_\beta^\pi\exp(tR\cos\theta)\,\mathrm{d}\theta \\
I_{22} &= AR^{1-\nu}\int_\beta^\pi\exp(tR\cos\theta-R^\nu\cos(\nu\theta)/A)\,\mathrm{d}\theta.
\end{aligned}
$$

Since $1-\nu > 0$, $\cos\beta = a/R$, $\lim_{R\to+\infty}\cos(\nu\beta) = \cos(\pi\nu/2) \neq 0$, and $\exp(tR\cos\theta - R^\nu\cos(\nu\theta)/A)$ is monotonic in $\theta \in [\beta,\pi]$ as $R \gg 1$, we obtain that

$$\begin{aligned}\lim_{R\to+\infty} I_{22} &= \lim_{R\to+\infty}\left[AR^{1-\nu}\int_\beta^\pi \exp(tR\cos\theta - R^\nu\cos(\nu\theta)/A)\,\mathrm{d}\theta\right]\\ &\leqslant A(\pi-\beta)\lim_{R\to+\infty}\left[R^{1-\nu}\exp(tR\cos\beta - R^\nu\cos(\nu\beta)/A)\right]\\ &= A(\pi-\beta)\mathrm{e}^{ta}\lim_{R\to+\infty}\left[R^{1-\nu}\exp(-R^\nu\cos(\nu\beta)/A)\right] = 0 \qquad (A.6)\end{aligned}$$

using L'Hospital rule. On the other hand,

$$\begin{aligned}\lim_{R\to+\infty} I_{21} &= \lim_{R\to+\infty}\left[AR^{1-\nu}\int_\beta^\pi \exp(tR\cos\theta)\,\mathrm{d}\theta\right]\\ &= A\lim_{R\to+\infty}\left[R^{1-\nu}\left(\int_\beta^{\pi/2}+\int_{\pi/2}^{\pi}\right)\exp(tR\cos\theta)\,\mathrm{d}\theta\right]. \qquad (A.7)\end{aligned}$$

Consider $\int_{\pi/2}^\pi \exp(tR\cos\theta)\,\mathrm{d}\theta$. Since $\exp(tR\cos\theta)$ is monotonic in θ, the theorem of integral mean value follows and there exists θ_R: $\pi/2 < \theta_R < \pi$ such that

$$\int_{\pi/2}^\pi \exp(tR\cos\theta)\,\mathrm{d}\theta = \frac{\pi}{2}\exp(tR\cos\theta_R) = \frac{\pi}{2}\exp(-tR|\cos\theta_R|).$$

Let $x_R = R|\cos\theta_R|$, then $0 < x_R < R$ and

$$\int_{\pi/2}^\pi \exp(tR\cos\theta)\,\mathrm{d}\theta = \frac{\pi}{2}\exp(-tx_R). \qquad (A.8)$$

Since $\int_{\pi/2}^\pi \exp(tR\cos\theta)$ is monotonically increasing in R and so x_R is decreasing. Let $x_\infty = \lim_{R\to\infty} x_R$, then $R \geqslant x_\infty > 0$. For any given $\varepsilon > 0$, when $R \gg 1$, $x_\infty > x_R > x_\infty - \varepsilon > 0$. Let $x_\infty - \varepsilon = R|\cos\theta_\varepsilon|$, $x_\infty = R|\cos\theta_\infty|$, it follows from (A.8) that

$$\frac{\pi}{2}\exp(-tR|\cos\theta_\infty|) \leqslant \int_{\pi/2}^\pi \exp(tR\cos\theta)\,\mathrm{d}\theta < \frac{\pi}{2}\exp(-tR|\cos\theta_\varepsilon|). \quad (A.9)$$

Using the L'Hospital rule we have

$$\lim_{R\to\infty} R^{1-\nu}\int_{\pi/2}^\pi \exp(tR\cos\theta)\,\mathrm{d}\theta \leqslant \lim_{R\to\infty}\frac{\pi}{2}[R^{1-\nu}\exp(-tR|\cos\theta_\varepsilon|)] = 0 \qquad (A.10)$$

when $1-\nu > 0$. It follows from (A.8)–(A.10) that

$$A\lim_{R\to+\infty}\left[R^{1-\nu}\int_{\pi/2}^\pi \exp(tR\cos\theta)\,\mathrm{d}\theta\right] = 0. \qquad (A.11)$$

From the monotonicity of $\exp(tR\cos\theta)$ it follows that

$$\begin{aligned}\int_\beta^{\pi/2} \exp(tR\cos\theta)\,\mathrm{d}\theta &< (\pi/2-\beta)\exp(tR\cos\beta)\\ &= (\pi/2 - \cos^{-1}(a/R))\exp(ta)\\ &= \mathrm{e}^{ta}\left[x(1-x^2)^{-1/2} + \tfrac{1}{2}x^3(1-x^2)^{-3/2} + \cdots\right] \quad (x = a/R)\\ &= \mathrm{e}^{ta}(a/R + \mathrm{o}(a/R)) \qquad (A.12)\end{aligned}$$

when $R \gg 1$. Thus,

$$\lim_{R\to+\infty}\left[R^{1-\nu}\int_\beta^{\pi/2}\exp(tR\cos\theta)\,\mathrm{d}\theta\right] \leqslant \lim_{R\to+\infty}\left[\mathrm{e}^{ta}R^{-\nu}(a + \mathrm{O}(a/R))\right] = 0. \qquad (A.13)$$

It follows from (A.7), (A.11) and (A.13) that

$$\lim_{R\to+\infty} I_{21} = \lim_{R\to+\infty}\left[AR^{1-\nu}\int_\beta^\pi \exp(tR\cos\theta)\,\mathrm{d}\theta\right] = 0. \qquad (A.14)$$

Therefore, it follows from (A.5), (A.6) and (A.14) that (A.4) holds when $0 < \nu < 1$.

Case II. $1 - \nu = 0$, i.e. $\nu = 1$.

It follows from (A.8) and (A.12) that

$$\lim_{R\to+\infty} I_{21} = \lim_{R\to+\infty} A \left(\int_{\beta}^{\pi/2} + \int_{\pi/2}^{\pi} \right) \exp(tR\cos\theta)\,\mathrm{d}\theta = 0. \tag{A.15}$$

Similarly, if $t > 1/A$, then

$$\lim_{R\to+\infty} I_{22} = \lim_{R\to+\infty} A \int_{\beta}^{\pi} \exp[(t - 1/A)R\cos\theta]\,\mathrm{d}\theta = 0. \tag{A.16}$$

Therefore, it follows from (A.4), (A.5), (A.15) and (A.16) that when $\nu = 1$ and $t > 1/A$,

$$\lim_{R\to+\infty} \left\{ I_2 = \int_{\Gamma_1} A\mathrm{e}^{pt} p^{-\nu} \left(1 - \exp(-p^{\nu}/A)\right)\,\mathrm{d}p \right\} = 0. \tag{A.17}$$

If $t = 1/A$, then

$$\lim_{R\to+\infty} I_{22} = A \lim_{R\to+\infty} (\pi - \beta) = A\pi/2 \neq 0. \tag{A.18}$$

If $t < 1/A$, i.e. $t - 1/A < 0$, then

$$\begin{aligned} \lim_{R\to+\infty} I_{22} = A \lim_{R\to+\infty} & \left\{ \int_{\beta}^{\pi/2} \exp\left[-(A^{-1} - t)R\cos\theta \right] \mathrm{d}\theta \right. \\ & \left. + \int_{\pi/2}^{\pi} \exp\left[(A^{-1} - t)R|\cos\theta| \right] \mathrm{d}\theta \right\} \\ = 0 + \lim_{R\to+\infty} & \int_{\pi/2}^{\pi} \exp\left[(A^{-1} - t)R|\cos\theta| \right] \mathrm{d}\theta \geqslant \pi/2 \neq 0. \end{aligned} \tag{A.19}$$

But $|I_2| \geqslant I_{21} - I_{22}$. Thus it follows from (A.15), (A.18) and (A.19) that, when $t = 1/A$ or $t < 1/A$ and $\nu = 1$, $\lim_{R\to+\infty} I_2 \neq 0$.

Case III. $\nu > 1$.

Since $\nu - 1 > 0$, it is obvious that

$$\lim_{R\to+\infty} I_{21} = A \lim_{R\to+\infty} \left[\frac{1}{R^{\nu-1}} \int_{\beta}^{\pi} \exp(tR\cos\theta)\,\mathrm{d}\theta \right] = 0.$$

Now, consider

$$\lim_{R\to+\infty} I_{22} = \lim_{R\to+\infty} \frac{A}{R^{\nu-1}} \int_{\beta}^{\pi} \exp(tR\cos\theta - R^{\nu}\cos(\nu\theta)/A)\,\mathrm{d}\theta.$$

For any given $0 < \eta < 1$, there exists an interval $[a_\nu, b_\nu] \subset (\beta, \pi)$ dependent on ν such that $\cos\nu\theta < 0$ and $|\cos\nu\theta| > \eta$. Thus

$$\begin{aligned} \lim_{R\to+\infty} I_{22} &\geqslant \lim_{R\to+\infty} \frac{A}{R^{\nu-1}} \int_{[a_\nu, b_\nu]} \exp(tR\cos\theta - R^{\nu}\cos(\nu\theta)/A)\,\mathrm{d}\theta \\ &= \lim_{R\to+\infty} \exp(-R^{\nu}\cos(\nu\theta)/A) \int_{[a_\nu, b_\nu]} \exp(tR\cos\theta)\,\mathrm{d}\theta \\ &\geqslant \lim_{R\to+\infty} \exp(-R^{\nu}\eta/A) \int_{[a_\nu, b_\nu]} \exp(tR\cos\theta)\,\mathrm{d}\theta = +\infty \end{aligned}$$

using the integral mean value theorem.

Thus, when $\nu > 1$, $\lim_{R\to+\infty} I_2 \neq 0$ because $|I_2| \geqslant I_{22} - I_{21}$.

From the above discussion we obtain the following result:

$$\lim_{R\to+\infty} \left\{ I_2 = \int_{\Gamma_1} A\mathrm{e}^{pt} p^{-\nu} \left(1 - \exp(-p^{\nu}/A)\right)\,\mathrm{d}p \right\} = 0 \tag{A.20}$$

if and only if $0 < \nu < 1$.

Similarly,

$$\lim_{R\to+\infty}\left\{I_3 = \int_{\Gamma_2} A\mathrm{e}^{pt}p^{-\nu}\,(1-\exp(-p^{\nu}/A))\;\mathrm{d}p\right\} = 0 \qquad \text{(A.21)}$$

if and only if $0 < \nu < 1$.

Therefore, from (A.2), (A.3), (A.20) and (A.21) we obtain that, when $0 < \nu < 1$,

$$\begin{aligned}
\mathcal{L}^{-1}[\tilde{M}(p)] &= \mathcal{L}^{-1}\left[Ap^{-\nu}\,(1-\exp(-p^{\nu}/A))\right] \\
&= -\lim_{\substack{R\to+\infty\\ \varepsilon\to+0}} \frac{1}{2\pi\mathrm{i}}\int_{C_1+C_2} A\mathrm{e}^{pt}p^{-\nu}\,(1-\exp(-p^{\nu}/A))\;\mathrm{d}p \\
&= -\frac{A}{2\pi\mathrm{i}}\left[-\int_0^{\infty}\frac{\mathrm{e}^{-tx}(1-\exp(-x^{\nu}\mathrm{e}^{\mathrm{i}\nu\pi}/A))}{x^{\nu}\exp(\mathrm{i}\nu\pi)}\,\mathrm{d}x\right. \\
&\quad \left. + \int_0^{\infty}\frac{\mathrm{e}^{-tx}(1-\exp(-x^{\nu}\mathrm{e}^{-\mathrm{i}\nu\pi}/A))}{x^{\nu}\exp(-\mathrm{i}\nu\pi)}\,\mathrm{d}x\right] \\
&= \frac{A}{\pi}(\sin\nu\pi)\int_0^{\infty}\mathrm{e}^{-tx}/x^{\nu}\,\mathrm{d}x = \frac{A\sin\nu\pi}{\pi}t^{\nu-1}\int_0^{\infty}\mathrm{e}^{-y}/y^{\nu}\,\mathrm{d}y \\
&= \frac{A\Gamma(1-\nu)\sin\nu\pi}{\pi}t^{\nu-1}.
\end{aligned}$$

References

[1] Havlin S and Ben-Avraham D 1987 *Adv. Phys.* **36** 695
[2] O'shaughnessy B and Procaccia I 1985 *Phys. Rev.* A **32** 3073–82
[3] O'shaughnessy B and Procaccia I 1985 *Phys. Rev. Lett.* **54** L1043
[4] Havlin S, Trus B and Weiss G H 1985 *J. Phys. A: Math. Gen.* **18** L1043
[5] Brooks Harris A and Aharony A 1987 *Europhys. Lett.* **4** 1355
[6] Bunde A and Havlin S (ed) 1991 *Fractals and Disordered Systems* (Heidelberg: Spinger)
[7] Giona M and Roman H E 1992 *J. Phys. A: Math. Gen.* **185** 87–97
[8] Roman H E 1997 *Fractals* **5** 379–93
[9] Ren F-Y, Liang J-Y and Wang X-T 1999 *Phys. Lett.* A **252** 141–50
[10] Ren F-Y, Yu Z-G and Su F 1996 *Phys. Lett.* A **219** 59–69
[11] Yu Z-G and Ren F-Y 1997 *J. Phys. A: Math. Gen.* **30** 5569–77
[12] Qiu W-Y and Lü J 2000 *Phys. Lett.* A **272** 353–8
[13] Ren F-Y, Yu Z-G and Zhou J 1999 *Southeast Asian Bull. Math.* **23** 497–505
[14] Devaney R L and Keen L 1989 *Chaos and Fractals (Proc. Symposia in Applied Mathematics* vol 39) pp 127–44